THE THEORY OF POLES AND POLARS.

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In the text-books on analytic geometry the symmetry of the equation

$$\Sigma a_{ij}x_i x_j' = 0 \quad [P]$$

is used to show that [P] is the equation of the chord (plane) of contact of the point x' with regard to the conic (quadric)

$$Q = \Sigma a_{ij}x_i x_j = 0,$$

while the theorems relating to secant lines (planes) are usually deferred until projective properties are taken up—usually in a separate course.

In addition to necessitating a preliminary treatment of tangent lines and planes, this method has the inconvenience that it does not give the properties which make ruled constructions of the tangents possible.

It is the purpose of this note to show that the various geometric properties given by the vanishing of the bilinear covariant,

$$\Sigma a_{ij}x_i x_j' = 0,$$

can be deduced in an elementary and simple manner from the equation itself, and in a way that is applicable without modification to quadratic loci in any number of dimensions.

The treatment here given is for quadrics in three dimensions and the coördinates used are point coördinates, though the processes admit, of course, the usual dual interpretation.

1. Writing our quadric in the form

$$\begin{aligned} Q &= \sum_1^4 a_{ij}x_i x_j \\ &\equiv (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4)x_1 + (a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4)x_2 \\ &\quad + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4)x_3 + (a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4)x_4 \\ &\equiv A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 0, \end{aligned}$$

where $a_{ij} = a_{ji}$ and $A_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4$. Thus $\Sigma a_{ij}x_i x_j' = 0$ is obtained by priming either the x's inside or the x's outside the parentheses.

A *vertex* is a set of x's which satisfy the four equations

$$A_i = 0, \ i = 1, 2, 3, 4, \tag{1}$$

and the quadric is non-singular if it has no vertex (system (1) of rank 4), a *cone* if it has one vertex (system (1) of rank 3), a *plane-pair* if it has a line of vertices (system (1) of rank 2), a double plane if it has a plane of vertices (system (1) of

rank 1). From this we have that *P vanishes identically if x' is a vertex, and passes through all the vertices if x' is not a vertex.*

2. Consider now the pencil of quadrics

$$\bar{Q} = k_1Q + k_2Q_1 = 0, \tag{2}$$

where

where $Q_1 = \Sigma b_{ij}x_ix_j$, and the polar of $\bar{Q}$ with respect to x' is

$$k_1\Sigma a_{ij}x_ix_j' + k_2\Sigma b_{ij}x_ix_j' = 0, \tag{3}$$

which is the equation of a pencil of planes. If $\bar{Q} = 0$ is a singular quadric of the pencil (2) and x' is one of its vertices, [3] vanishes identically, so that

$$P \equiv k_1\Sigma a_{ij}x_ix_j' \equiv - k_2\Sigma b_{ij}x_ix_j' = 0.$$

Theorem: *thus we have that the polars of x' with respect to all the quadrics of pencil* (2) *are the same, if x' is a vertex of a quadric of the pencil. Moreover, P will in this case pass through the vertices of all other singular quadrics of the pencil.*

3. The application of this theorem leads at once to the geometric properties in question.

First let Q_1 be two planes, Q_1 is then a singular quadric of the pencil with a line of vertices which we shall suppose passes through the point x'. The polar of x' thus passes through the vertices of all the *other* singular quadrics of the pencil, *i. e.*, through the vertices of the two cones passing through the two conics cut out of $Q = 0$ by $Q_1 = 0$. If now one of the planes of which $Q_1 = 0$ consists becomes tangent to $Q = 0$ the vertex of one of these cones will become the point of contact of this tangent plane, so that $P = 0$ is the equation of the plane of contact of x'. If now the point x' approach the surface of $Q = 0$ as limit, $P = 0$ approaches the tangent plane so that if x' is on $Q = 0$, $P = 0$ is the tangent plane at x'.

Furthermore, if x' is any point on the line of vertices of $Q_1 = 0$,

$$P = \Sigma a_{ij}x_i(k_1x_j' + k_2x_j'') = 0$$

will determine a pencil whose axis,

$$\Sigma a_{ij}x_ix_j' = 0, \Sigma a_{ij}x_ix_j'' = 0,$$

is the conjugate line of $(x'x'')$.

Secondly, if Q_1 is a double plane, its pole lies in P. More generally if $Q_1 = 0$ is any cone with vertex at x', $P = 0$ passes through the vertices of the three remaining cones which pass through the twisted quartic in which $Q = 0$ and $Q_1 = 0$ intersect. The existence of the remaining cones is shown, of course, by setting the discriminant of $\bar{Q}$ equal to zero.